Graph Theory

Introduction

Graph theory is the study of graphs i.e. structures which are used to model relationships between objects. These can be as straightforward as paths or roads connecting towns and cities. Or they can be used in more unconventional ways such as modelling the connection between atoms in molecules, or analysing connections between people on social networks like Facebook.

Aim of the Workshop

The aim of this workshop is to introduce students to the basic concepts of graph theory through exploring the Bridges of Königsberg problem. The students will attempt to solve the problem before being introduced to the theory behind the solution and the solution to the problem itself. The workshop will also give insights into how graph theory is applied in various disciplines.

Learning Outcomes

By the end of this workshop students should be able to:

- Represent problems in graphical form
- Define the conditions required for an Euler path to exist
- Recognise the Hand–Shaking Lemma and the 4 Colour Theorem and describe their relevance to Graph Theory
- Recognise other applications of Graph Theory.

Materials and Resources

Each student will require: paper, pens, activity sheets (activity sheet 1 preferably laminated), whiteboard marker, tissues/erasers,

Key Words

Graph
structures which are used to model relationships between objects.

Bridges of Konigsberg
A problem which is credited as the beginnings of Graph Theory derived by Leonhard Euler.

Euler/ Euler Path
Euler was a Swiss mathematician, physicist, astronomer and engineer. He presented a solution to the Bridges of Konigsberg problem in 1735 leading to the definition of an Euler Path, a path that went over each road exactly once.

Vertex/ Edges
See Appendix – Note 1.

Graph Theory: Workshop Outline

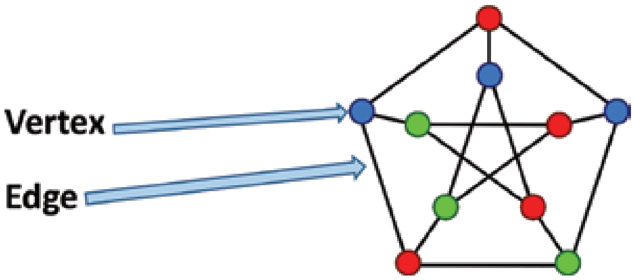
SUGGESTED TIME (TOTAL MINS)	ACTIVITY	DESCRIPTION
5 mins (00:05)	**Introduction to workshop and Graph Theory**	– Ask students what they think graph theory might involve. Highlight the uses of graph theory in social networks such as Facebook etc. – Students should be divided into groups of 5
5 mins (00:10)	**The Bridges of Konigsberg Problem**	– Introduce the Bridges of Konigsberg problem: "People wondered if they could visit all areas of the city while only crossing each bridge exactly once" – **Activity Sheet 1:** Can you walk through the town crossing each bridge exactly once? – Students should try to trace such a path on the laminated maps (or in pencil if not laminated).
10 mins (00:20)	**Representing the problem as a Graph**	– Introduce the concepts of graphs by trying to simplify the picture of the town, with each land mass as a point and the bridges as lines connecting the pints as in the image below: – **Activity Sheet 2:** Find paths for simple shapes. Students should work together to decide if they can find a path that goes over each line once (draw on **Sheet 2A**, answers on **Sheet 2B**)
10 mins (00:30)	**Elementary Graph Theory**	– Introduce the components of a graph – **Vertices** and **edges**, **odd** and **even** degree for vertices (see **Note 1**) Degree of a vertex = number of paths entering that vertex. – **Activity Sheet 3:** Calculate the number of vertices for each of the simple graphs from before and identify the number of vertices of odd degree and even degree.

SUGGESTED TIME (TOTAL MINS)	ACTIVITY	DESCRIPTION
10 mins (00:40)	**Solution to the Bridges of Konigsberg Problem**	– Discuss the mathematician Euler and the definition of an Euler path (a path that crosses each edge exactly once) – Ask students to compare which of the earlier shapes had paths to the number of vertices of odd degree. See if students can spot the rule for deciding if there is an Euler path. – Define that an Euler path exists only if there is exactly zero or two vertices of odd degree – Note that for no odd vertices you start and finish on the same point but for two odd vertices you must start on one odd vertex and finish on the other! (see **Appendix – Note 3**) – Reveal that the Bridges of Konigsberg problem has no solution! – **Activity Sheet 4:** Which of the following (new) graphs have Euler Paths? Students should solve this using the new theory they have learned rather than tracing out the path
10 mins (00:50)	**The Hand–shaking Lemma**	– **Activity Sheet 5:** Students shake hands with everyone in their group – Students are asked to calculate the number of hands they shook and the number of handshakes that occurred in total – Students are asked to try draw this problem as a graph – **HINT:** let each person be a vertex and each handshake be an edge!
10 mins (01:00)	**The four colour theorem**	– Show the students an image of South America and pose the question: "If two bordering countries cannot be the same colour what is the fewest number of colours we need to colour in the entire continent?" – Students attempt colouring the map **(Activity Sheet 6)** – Ask students "Can we draw a graph to work out the minimum number of colours needed?" – (Note: there are only 4 colours needed)
10 mins (01:10)	**Hamilton Paths and the Icosian game**	– Task: **Activity Sheet 2** In pairs, students work together using backwards induction (starting with 2 pirates and working up to 5 pirates) to determine the optimal suggestion for the eldest pirate – Whole class discussion on strategies and solution (as noted in **Appendix – Note 4**)

Graph Theory Appendix:

Note 1: Graph theory definitions

Degree is the number of edges entering or leaving the vertex



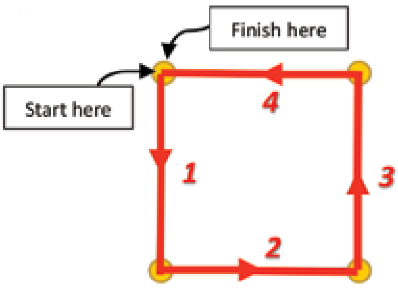
Note 2: Graph Theory Slides

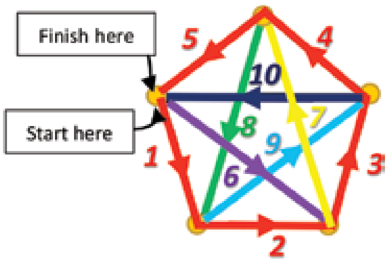
Slides for this workshop can be found at the following address:
https://prezi.com/njwia55925qs/graph–theory/?utm_campaign=share&utm_medium=copy

Please note that these slides do not include the four colour theorem or a discussion of Activity Sheet 5 or 6.

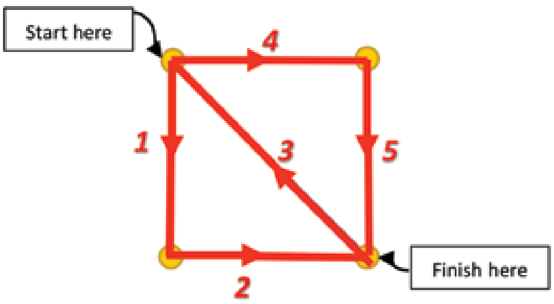
Note 3: Examples of zero odd vertices and two odd vertices

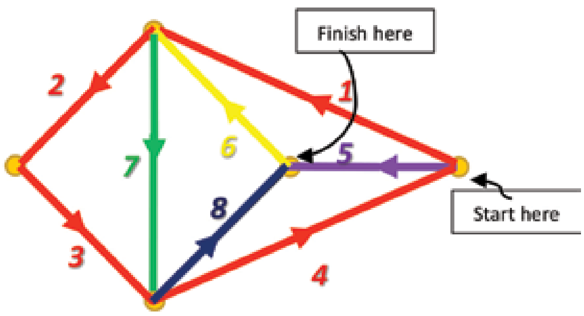
If a graph has **no** vertices of odd degree then you must *start and finish at the same vertex* to complete an Euler path



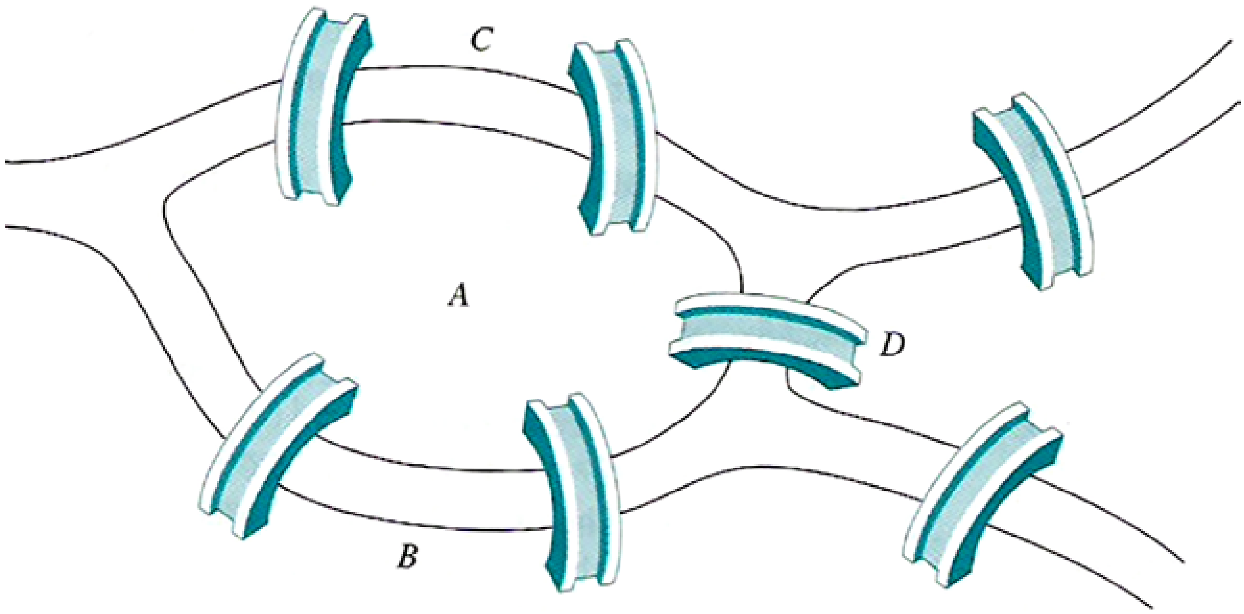


If a graph has **two** vertices of odd degree then you *must start at one of the odd vertices and finish at the other odd vertex* to complete an Euler path

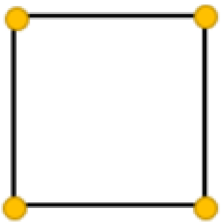




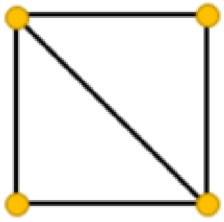
Graph Theory – Activity Sheet 1



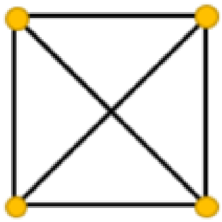
Graph Theory – Activity Sheet 2A



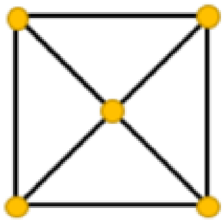
1



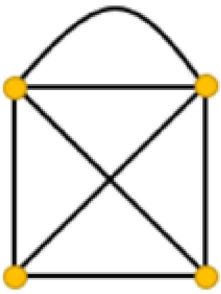
2



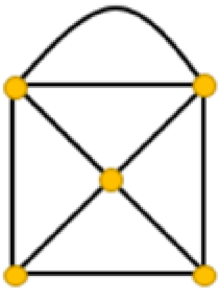
3



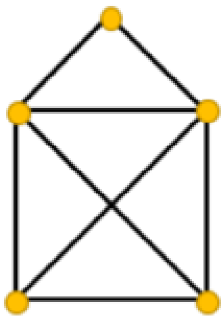
4



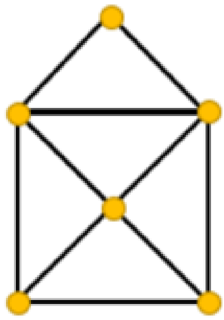
5



6

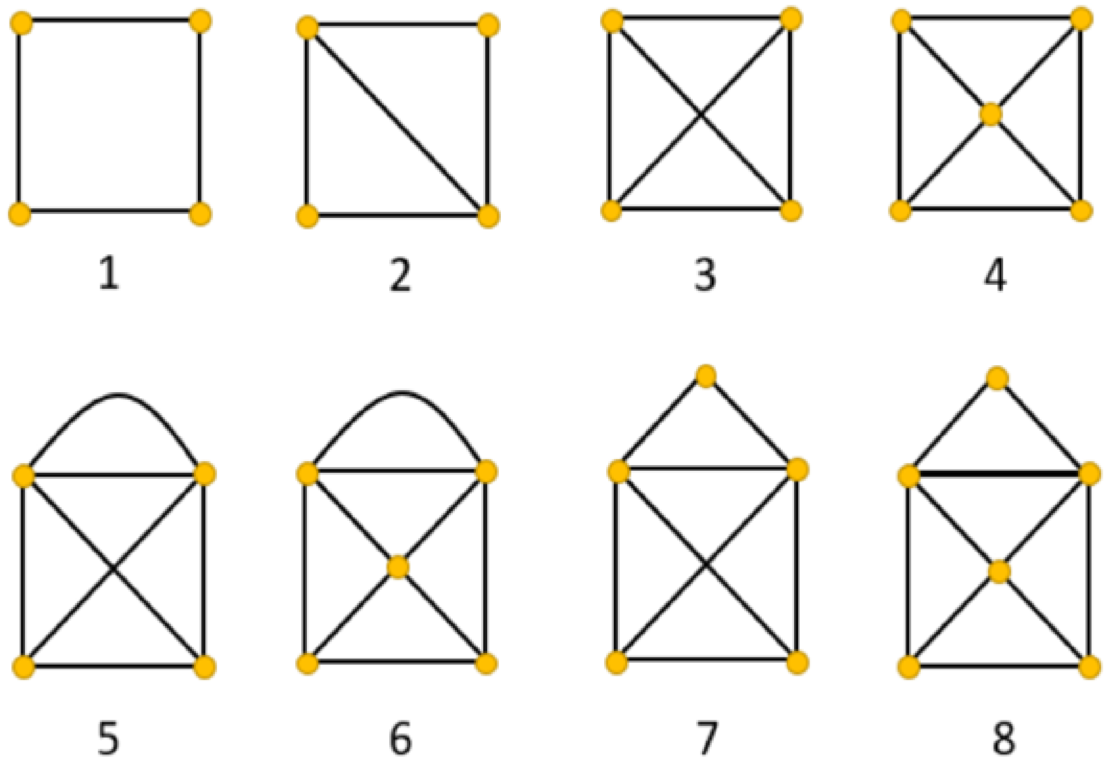


7



8

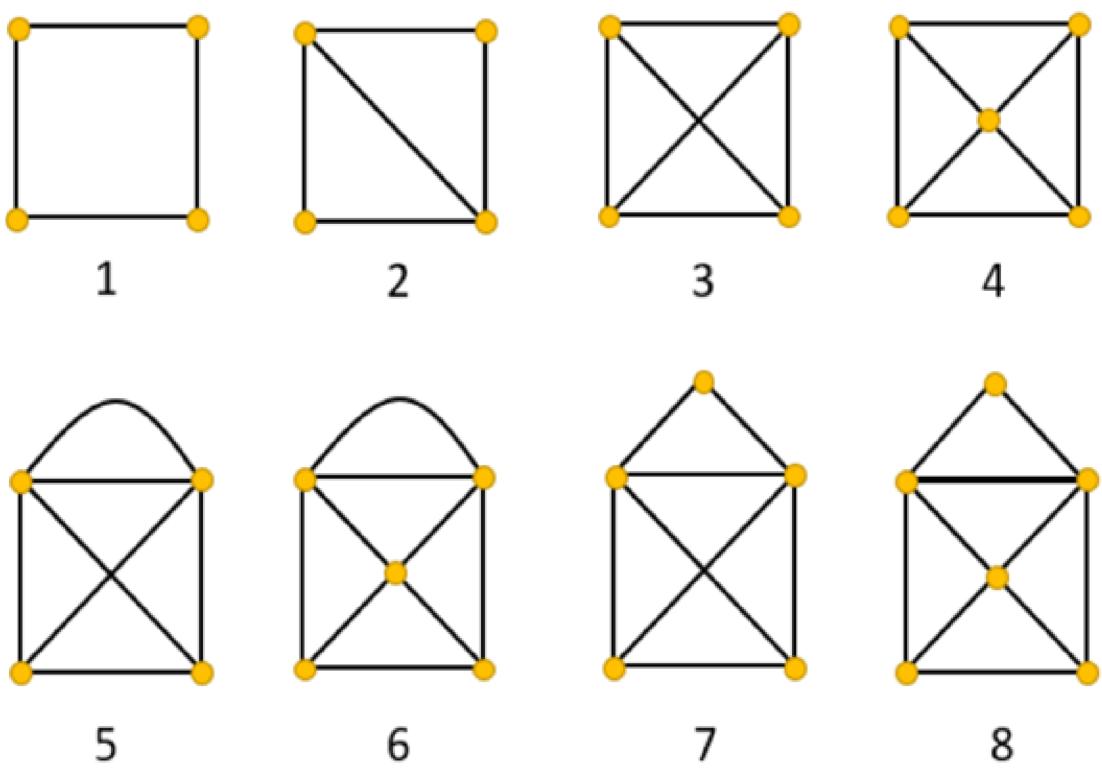
Graph Theory – Activity Sheet 2B



GRAPH	PATH; YES/NO
1	
2	
3	
4	
5	
6	
7	
8	

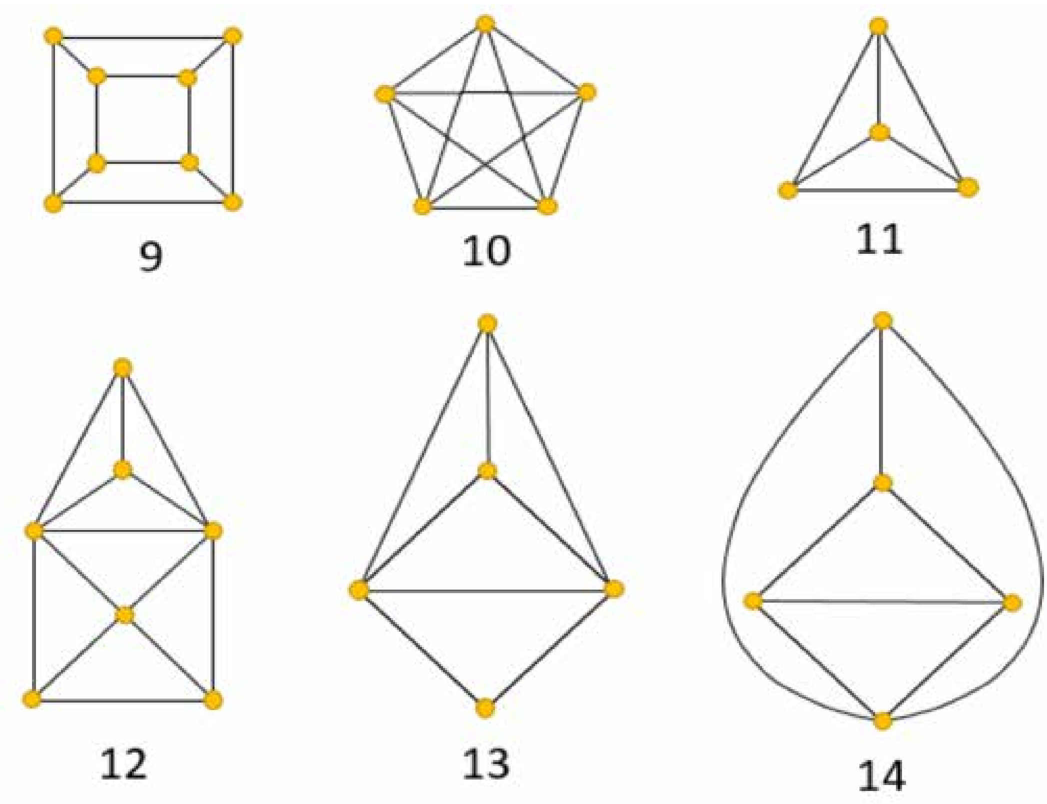
Hint – There are two which do not have a path!

Graph Theory – Activity Sheet 3



GRAPH	NUMBER OF VERTICES	NUMBER WITH EVEN DEGREE	NUMBER WITH ODD DEGREE	PATH; YES/NO
1				
2				
3				
4				
5				
6				
7				
8				

Graph Theory – Activity Sheet 4



GRAPH	NUMBER OF VERTICES	NUMBER WITH EVEN DEGREE	NUMBER WITH ODD DEGREE	PATH; YES/NO
9				
10				
11				
12				
13				
14				

Graph Theory – Activity Sheet 5

"Everyone shake hands with each person in their group..."

How many hands did you shake?

How many handshakes in total?

Can you draw a graph to represent the handshakes?

Graph Theory – Activity Sheet 6

What is the minimum number of colours needed to colour the map below, so that neighbouring countries do not have the same colour?

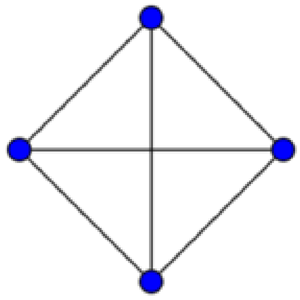
Credit: http://www.printablemaps.net/south–america–maps/

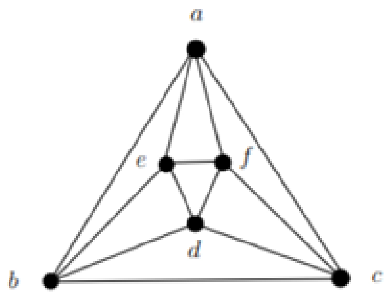
Can you draw a graph to work out the answer to the above question?

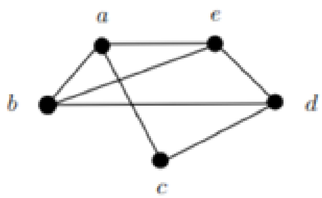
HINT: think about what a vertex or edge might represent on your graph

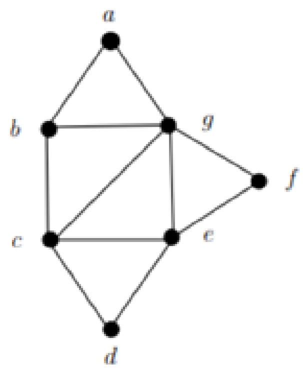
Graph Theory – Activity Sheet 7

1. **Decide whether the graphs below have Hamiltonian paths or Hamiltonian cycles or none of them.**

a) 

b) 

c) 

d) 

HINT: A Hamilton Cycle is a Hamilton path that begins and ends at the same vertex

2. **Try to find a Hamiltonian cycle in Hamilton's famous Icosian game.**